Delta Conjecture and affine Springer fibers

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Outline

Choose two numbers $k \leq n$, define $K = k(n-k+1)$. In this talk, I will explain various ingredients in the following table:

	Degree	Algebra	Combinatorics	Geometry	Module
(1)	K	$E_{K,k} \cdot 1$	$\mathrm{PF}_{K,k}$	$Y_{n,k}$	$H_*^{BM}(Y_{n,k}) \circlearrowleft S_K$
(2)	n	$\Delta'_{e_{k-1}} e_n$	$\mathcal{LD}_{n,k}^{\mathrm{stack}}$	$X_{n,k}$	$H_*^{BM}(X_{n,k}) \circlearrowleft S_n$

The row (1) presents a symmetric function of degree K which is related to **Compositional Rational Shuffle Conjecture**. This symmetric function also appears as a character of the S_K action in the Borel-Moore homology of some space $Y_{n,k}$.

Similarly, the row (2) presents a symmetric function of degree n which is related to **Delta Conjecture**, and the homology of another space $X_{n,k}$.

Outline

Theorem (Gillespie-G.-Griffin)

All symmetric functions in row (1) agree.
All symmetric functions in row (2) agree.

Theorem (Gillespie-G.-Griffin)

We have $s_\lambda^\perp(1) = (2)$ *where* $\lambda = (k-1)^{n-k}$ *is the rectangular Young diagram, and* $s_\lambda^\perp$ *is adjoint to multiplication by* s_λ *with respect to the Hall inner product.*

Note that the operator $s_\lambda^\perp$ decreases the degree by $(k-1)(n-k) = K - n$.

Shuffle Conjecture

The case $k = n$ of the table corresponds to the celebrated **Shuffle Conjecture** proposed by Haiman, Haglund, Loehr, Remmel and Ulyanov, and first proved by Carlsson and Mellit.
On the **algebraic side** we have the symmetric function ∇e_n where e_n is the elementary symmetric function and ∇ is the operator which diagonalizes in the basis of **modified Macdonald polynomials**:

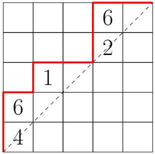
$$\nabla \widetilde{H}_\lambda = q^{n(\lambda)} t^{n(\lambda')} \widetilde{H}_\lambda.$$

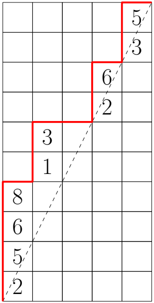
Here $q^{n(\lambda)} t^{n(\lambda')}$ is the product of (q,t)-contents of boxes in the diagram of λ. For example, $\nabla \widetilde{H}_{3,2} = q^4 t^2 \widetilde{H}_{3,2}$.

t	qt	
1	q	q^2

Shuffle Conjecture

On the combinatorial side we consider Dyck paths in the $n \times n$ square that stay weakly above the northeast diagonal in the grid. A **word parking function** is a labeling of the vertical runs of the Dyck path by positive integers such that the labeling strictly increases up each vertical run (but letters may repeat between columns; hence "word" parking function). We let $\mathrm{WPF}_{n,n}$ be the set of word parking functions.





Later we will consider word parking functions in arbitrary rectangles, see right picture.

Shuffle Conjecture

Theorem (Shuffle Conjecture, Carlsson-Mellit)

We have

$$\nabla e_n = \sum_{P \in \mathrm{WPF}_{n,n}} t^{\mathrm{area}(P)} q^{\mathrm{dinv}(P)} x^P,$$

where area *and* dinv *are certain statistics on word parking functions.*

Theorem (Hikita, G.-Mazin-Vazirani)

There is an algebraic variety X_n with an action of S_n in the (Borel-Moore) homology of X_n such that:

a) X_n has an affine paving with cells in bijection with parking functions. The dimension of the cell equals dinv *of the parking function.*

b) The Frobenius character of $H_*(X_n)$ equals ∇e_n.

X_n is an example of **affine Springer fiber**, to be defined later in the talk.

Rational Shuffle Conjecture

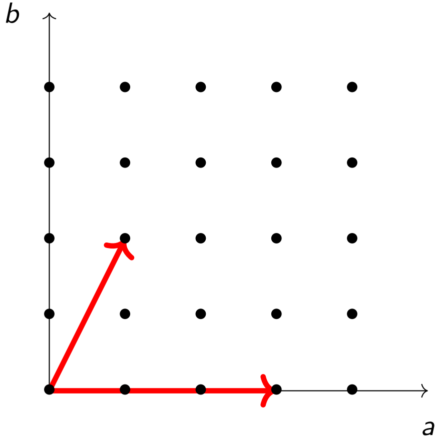
We will need a generalization of the Shuffle Theorem known as the **Compositional Rational Shuffle Theorem**, conjectured by Bergeron-Garsia-Leven-Xin and proved by Mellit. To state it, we need to recall some constructions related to the **Elliptic Hall Algebra** $\mathcal{E}_{q,t}$. The algebra $\mathcal{E}_{q,t}$ has generators $P_{a,b}, (a,b) \in \mathbb{Z}^2$ satisfying certain complicated relations. We will not need these relations but record some useful properties:

(a) If $(a', b') = (ca, cb) \in \mathbb{Z}^2$ for some rational constant $c > 0$, then $[P_{a,b}, P_{a',b'}] = 0$.

(b) A certain extension of the group $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathcal{E}_{q,t}$ by algebra automorphisms. If $M \in \mathrm{SL}(2,\mathbb{Z})$ then the corresponding automorphism sends the generator $P_{a,b}$ to $P_{M(a,b)}$, up to a certain monomial in q, t.

(c) The algebra $\mathcal{E}_{q,t}$ acts on $\Lambda(q,t)$. The operator $P_{a,b}$ has degree a, that is, $\deg P_{a,b}(f) = \deg f + a$. The operators $P_{a,0}$ act on $\Lambda(q,t)$ by multiplication by power sums p_a (up to a scalar factor).

The generators $P_{3,0}$ and $P_{1,2}$ of the Elliptic Hall Algebra:



Rational Shuffle Conjecture

Suppose $\gcd(a,b)=1$. Since $P_{ia,ib}$ pairwise commute for $i>0$, the algebra $\mathcal{E}_{q,t}$ has a large commutative subalgebra of slope b/a.

Given a symmetric function $F\in\Lambda(q,t)$, we can transform it to an operator $F_{b/a}$ in $\mathcal{E}_{q,t}$ as follows: first expand F in power sums p_i, then replace each p_i by $P_{ia,ib}\in\mathcal{E}_{q,t}$. Alternatively, we can find $M\in\mathrm{SL}(2,\mathbb{Z})$ such that $M(1,0)=(a,b)$, then the corresponding automorphism of $\mathcal{E}_{q,t}$ sends F (thought of as a multiplication operator and hence an element of $\mathcal{E}_{q,t}$ of slope zero) to $F_{b/a}$.

Suppose $\gcd(a,b)=d$. We define the operator $E_{a,b}\in\mathcal{E}_{q,t}$ as the result of rotation of the elementary symmetric function e_d to slope b/a as above.

Rational Shuffle Conjecture

On the combinatorial side of the Compositional Rational Shuffle Conjecture we have the sum over rational parking functions in the $a \times b$ rectangle:

Theorem (Mellit)

$$E_{a,b} \cdot 1 = \sum_{P \in \mathrm{WPF}_{a,b}} q^{\mathrm{area}(P)} t^{\mathrm{dinv}(P)} x^P.$$

Example

For $a = b = n$ one can check that $E_{a,b} \cdot 1 = \nabla e_n$, and we recover Shuffle Theorem.

If $\gcd(a, b) > 1$, Mellit proved a more general result where the touch points of the Dyck path and the diagonal are specified. We will not need it.

Delta Conjecture

Next, we discuss **Delta Conjecture**, proposed by Haglund, Remmel and Wilson, and recently proved by Blasiak-Haiman-Morse-Pun-Seelinger, and D'Adderio-Mellit.

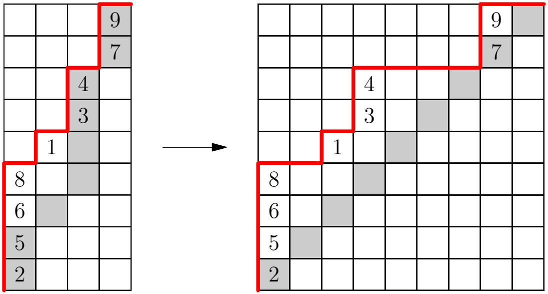
On the algebraic side we have the symmetric function $\Delta'_{e_{k-1}} e_n$ where $\Delta'_{e_{k-1}}$ is the operator on $\Lambda(q,t)$ which is diagonal in the modified Macdonald basis $\widetilde{H}_\lambda$ with eigenvalues

$$\Delta'_{e_{k-1}} \widetilde{H}_\lambda = e_{k-1}[B'_\lambda] \widetilde{H}_\lambda, \quad B'_\lambda = \sum_{\square \in \lambda,\, \square \neq (0,0)} q^{a'(\square)} t^{\ell'(\square)}.$$

Delta Conjecture

On the combinatorial side we have **stacked parking functions**. A **stack** S of boxes in an $n \times k$ grid is a subset of the grid boxes such that there is one element of S in each row, at least one in each column, and each box in S is weakly to the right of the one below it.

A (word) **stacked parking function** with respect to S is a labeled up-right path D such that each box of S lies below D, and the labeling is strictly increasing up each column.



Delta Conjecture

Theorem (Delta Conjecture)

$$\Delta'_{e_{k-1}} e_n = \sum_{P \in \mathcal{LD}^{\mathrm{stack}}_{n,k}} q^{\mathrm{area}(P)} t^{\mathrm{hdinv}(P)} x^P.$$

Theorem (Gillespie-G.-Griffin)

Letting $K = k(n-k+1)$ and $\lambda = (k-1)^{n-k}$, we have

$$\Delta'_{e_{k-1}} e_n = s_\lambda^\perp (E_{K,k} \cdot 1), \tag{1}$$

where $s_\lambda^\perp$ is the adjoint to multiplication by the Schur function s_λ.

We give two proofs of thsi theorem, by applying $s_\lambda^\perp$ both to the combinatorial and algebraic sides of the Compositional Rational Shuffle Theorem for (K,k). As a consequence, we obtain a new proof of Delta Conjecture.

Springer fibers

Next, we switch to geometry. Recall that the complete flag variety in $\mathbb{C}^K$ is defined as the space of flags

$$\{0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_K = \mathbb{C}^K : \dim V_i = i\}.$$

Let γ be a nilpotent operator on $\mathbb{C}^K$ with Jordan type μ. The **Springer fiber** Z_γ is defined as the space of flags such that $\gamma V_i \subset V_i$ for all i.

Theorem (Hotta-Springer)

There is an S_K action in the (Borel-Moore) homology of Z_γ. Its Frobenius character equals the **Hall-Littlewood polynomial** *$H_\mu(x; q)$.*

Springer fibers

We can also consider the partial flag variety:

$$\{V_\bullet : 0 = V_0 \subset V_1 \subset V_2 \subset \dots V_n \subset V_K = \mathbb{C}^K : \dim V_i = i\}.$$

Given γ as above, we can consider the "Δ-Springer variety" (defined by Griffin-Levinson-Woo):

$$W_{\mu,k,n} = \{V_\bullet : \gamma V_i \subset V_i, \mathrm{JT}\left(\gamma|_{V_K/V_n}\right) \leq (n-k)^{k-1}\}.$$

Theorem (Gillespie-Griffin)

There is an S_n action in in the (Borel-Moore) homology of $W_{\mu,k,n}$. Under some assumption on μ, the Frobenius character of this action equals $s_\lambda^\perp H_\mu(x;q)$, where $\lambda = (n-k)^{k-1}$ as above.

Affine Springer fibers

We generalize the above results to the affine flag variety. We will work with the ring of power series $\mathbb{C}[[\epsilon]]$ and the field of Laurent series $\mathbb{C}((\epsilon))$.

A lattice $\Lambda \subset \mathbb{C}^K((\epsilon))$ is a free $\mathbb{C}[[\epsilon]]$-submodule of rank K. The **affine flag variety** is the space of flags of lattices:

$$\{\Lambda_\bullet : \Lambda_1 \supset \Lambda_2 \supset \cdots \supset \Lambda_K \supset \epsilon\Lambda_1.\}$$

The affine flag variety is the union of affine Schubert cells labeled by the affine permutations in $\widetilde{S_K}$.

Given $\gamma \in \mathrm{GL}_K((\epsilon))$, the **affine Springer fiber** Sp_γ is defined as the space of flags of lattices as above such that $\gamma\Lambda_i \subset \Lambda_i$.

Affine Springer fibers

We define the space $Y_{n,k} = \mathrm{Sp}_\gamma \cap C$ where γ is a certain explicit operator (depending on n, k) and C is the union of certain affine Schubert cells,

Theorem (Gillespie-G.-Griffin)

a) The space $Y_{n,k}$ has an affine cell decomposition with cells in bijection with (K, k) rational parking functions.
b) The dimension of the cell is equal (up to a constant) to the dinv statistic of the parking function.
c) There is an S_K action in the Borel-Moore homology of $Y_{n,k}$, and the corresponding Frobenius character equals $E_{K,k} \cdot 1$.

For experts, the main difficulty here (and the reason to introduce C) is that K and k are not coprime (in fact, k divides K).

Affine Springer fibers

Finally, we define the "affine Borho-Macpherson space" $BM_{n,k}$ as the space of partial flags of lattices

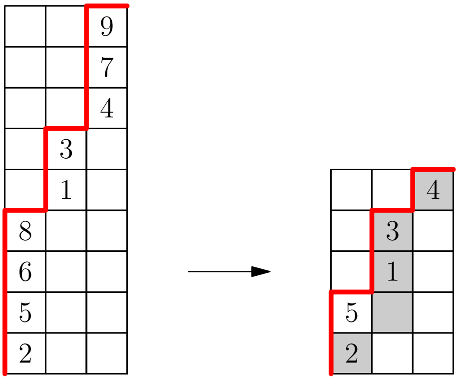
$$\{\Lambda_\bullet : \Lambda_1 \supset \Lambda_2 \supset \cdots \Lambda_n \supset \Lambda_K \supset \epsilon\Lambda_1 .\}$$

such that $\gamma\Lambda_i \subset \Lambda_i$ and $\mathrm{JT}\left(\gamma|_{\Lambda_n/\Lambda_K}\right) \leq (n-k)^{k-1}$. We define $X_{n,k} = BM_{n,k} \cap C'$ where C' is another union of affine Schubert cells.

Theorem (Gillespie-G.-Griffin)

There is an S_n action in the Borel-Moore homology of $X_{n,k}$, and the corresponding Frobenius character equals $\Delta'_{e_{k-1}} e_n$.

Appendix: from (K, k) parking functions to stacked parking functions



Here $n = 5, k = 3$, so $K = k(n - k + 1) = 9$. We split the labels into "big" $(> n)$ and "small" $(\leq n)$. Erase the big labels, the result is a labeled path in $k \times n$ rectangle. The stack remembers how many boxes we erased in each column.

Thank You!